

Holomorphic vector bundles on primary Kodaira surfaces

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Introduction

Let X be a smooth compact complex surface. A classical problem is to decide which topological complex vector bundles admit holomorphic structures, or equivalently, to find all the triples $(r, c_1, c_2) \in \mathbf{N} \times NS(X) \times \mathbf{Z}$, $r \geq 2$ for which there exists a rank r holomorphic vector bundle \mathcal{E} on X with Chern classes $c_1(\mathcal{E}) = c_1$ and $c_2(\mathcal{E}) = c_2$.

For projective surfaces, Schwarzenberger (cf. [Schw]) proved that any triple $(r, c_1, c_2) \in \mathbf{N} \times NS(X) \times \mathbf{Z}$, $r \geq 2$ comes from a rank r holomorphic vector bundle. In contrast to this situation, for non-projective surfaces, there is a natural *necessary* condition for the existence problem (cf. [BaL] Theorem 3.1 for the general case; for the rank-2 case see [BrF] Proposition 1.1, [ElFo]):

$$\Delta(r, c_1, c_2) := \frac{1}{r} \left(c_2 - \frac{r-1}{2r} c_1^2 \right) \geq 0.$$

Using essentially extensions of coherent sheaves one proves the following *sufficient* condition for the existence of holomorphic vector bundles (cf [BaL]; see also [BrF], [LeP]):

$$\Delta(r, c_1, c_2) \geq m(r, c_1),$$

where

$$m(r, c_1) := -\frac{1}{2r} \max \left\{ \sum_{i=1}^r \left(\frac{c_1}{r} - \mu_i \right)^2, \mu_1, \dots, \mu_r \in NS(X), \sum_{i=1}^r \mu_i = c_1 \right\},$$

with the sole excepted case: X a $K3$ surface with algebraic dimension zero, c_1 divisible by r in $NS(X)$ and $\Delta(r, c_1, c_2) = \frac{1}{r}$. This result also has a converse, which is (see [BaL], [BrF], [LeP]) any *filtrable* rank r holomorphic vector bundle \mathcal{E} on a non-algebraic surface X with Chern classes $c_1(\mathcal{E}) = c_1$ and $c_2(\mathcal{E}) = c_2$ satisfies the inequality

$$\Delta(\mathcal{E}) := \Delta(r, c_1, c_2) \geq m(r, c_1).$$

Therefore the only unknown situations are in the range $\Delta(r, c_1, c_2) \in [0, m(r, c_1))$. If $m(r, c_1) \neq 0$, this interval is non-empty, and in order to solve the existence problem, one has to construct holomorphic vector bundles having the corresponding

discriminant Δ lower than $m(r, c_1)$. Of course, all these vector bundles will be *non-filtrable* and the difficulty of the problem resides in the lack of a general construction method in this case (for more details see, for example, [Bri3]). One is compelled therefore to focus on particular classes of surfaces and find some specific construction methods.

The main result in the present paper is the following:

Theorem. *Let X be a primary Kodaira surface. Then for a triple $(r, c_1, c_2) \in \mathbf{N} \times NS(X) \times \mathbf{Z}$, with $r \geq 2$, there exists a holomorphic rank- r vector bundle \mathcal{E} on X such that $c_1(\mathcal{E}) = c_1$ and $c_2(\mathcal{E}) = c_2$ if and only if $\Delta(r, c_1, c_2) \geq 0$.*

From now on by a *vector bundle* we shall mean a *holomorphic vector bundle*, and a *curve* will always be a smooth, complex projective curve.

Recall that a primary Kodaira surface X is a principal elliptic bundle over some elliptic curve B . The case when $r\Delta(r, c_1, c_2)$ is an integer has been previously solved (cf. [To2]; see also [To1]) by using *unramified* coverings of B by another suitable elliptic curve and suitable deformations of sheaves. For the remaining cases, we use *curves of genus two with elliptic differentials*, i.e. curves C of genus 2 which admit a non-constant morphism to an elliptic curve, which does not factor over an isogeny of the elliptic curve. These curves were studied extensively by Bolza, Humbert, Picard, Poincaré and many others (see Chapter XI of Krazer's book [Kr]), and a remarkable thing about them is that the covers occur in pairs. More precisely (for details, see for example [FrKa]), if we consider $f : C \rightarrow B$ such a covering, and choose $F = \text{Ker}(J_C \rightarrow B)$, then there is a *complementary* natural covering $g : C \rightarrow F$ (induced by the embedding of C in J_C), giving rise to an isogeny of degree r^2 , $f^* \times g^* : B \times F \rightarrow J_C$. Its kernel H is the graph of an isomorphism $\psi : B[r] \rightarrow F[r]$ (which is *anti-isometric* with respect to the Weil pairings).

Conversely, one can get such curves out of a particular case of the "Basic Construction" and "Reducibility Criterion" (cf. [Ka2] and [We]; see also [Ka1], [FrKa]). Suppose one starts with B and F two elliptic curves, and $\Theta = 0 \times F + B \times 0$ the canonical principal polarization on $B \times F$ given by the product structure. Assume $\psi : B[r] \rightarrow F[r]$ is an isomorphism between the r -torsion subgroups, anti-isometric with respect to the Weil pairings. Then, by denoting $H_\psi = \text{Graph}(\psi)$ (which is a maximal isotropic subgroup of $(B \times F)[r]$), and $p : B \times F \rightarrow (B \times F)/H_\psi$ the canonical projection, it turns out that the surface $(B \times F)/H_\psi$ carries a principal polarization Θ_ψ such that $p^*\Theta_\psi$ is linear equivalent to $r\Theta$ (cf. [FrKa], [Ka2]; see also [Mumf1]). If the anti-isometry ψ is irreducible (cf. [We]; see [Ka2] for an analysis of this case), then the linear system $|\Theta_\psi|$ contains a smooth curve C of genus 2. In particular, the principally polarized abelian surface $((B \times F)/H_\psi, \Theta_\psi)$ is isomorphic to the Jacobian of C , and C covers r -to-one the curves B and F .

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1. Topological information on principal elliptic bundles

This section is devoted to some results which are used in the proof of the Theorem. We start by giving a formula for computing the degree of a covering of an elliptic curve.

Lemma 1.1. *Let $\gamma : B \rightarrow E$ be a covering of an elliptic curve with a curve of genus g . Choose $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\} \in H_1(B, \mathbf{Z})$ a symplectic basis. If $\gamma_* : H_1(B, \mathbf{Z}) \rightarrow H_1(E, \mathbf{Z})$ denotes the natural push-forward map, then*

$$\deg(\gamma) = \sum_{i=1}^g \gamma_*(\alpha_i) \cdot \gamma_*(\beta_i).$$

Proof. Consider the Abel-Jacobi embedding of B in its Jacobian J_B , corresponding to a chosen point in B . Following classical notations (cf. [LanBi], p. 328), the class of B as a cycle in J_B is represented by $\{\widetilde{W}_1\} \in H_2(J_B, \mathbf{Z})$, $\{\widetilde{W}_1\} = -\sum_{i=1}^g \alpha_i \star \beta_i$, where " \star " denotes the Pontrjagin product. If we denote $V = H^0(B, K_B)^*$, then for any divisor D on J_B whose Chern class is represented, by means of Appell-Humbert theorem, by an alternating form $A : V \times V \rightarrow \mathbf{R}$, we have (cf. [LanBi], p.42-43, p. 104, p.106):

$$(1.1) \quad (\{\widetilde{W}_1\} \cdot \{D\}) = \sum_{i=1}^g A(\alpha_i, \beta_i).$$

If one considers now D a fiber of the norm map $Nm(\gamma) : J_B \rightarrow E$, then the associated alternating form is $A : V \times V \rightarrow \mathbf{R}$, defined by $A(u, v) = \gamma_*(u) \cdot \gamma_*(v)$ for all $u, v \in H_1(B, \mathbf{Z})$. Now apply the relation (1.1), and observe that $\deg(\gamma) = (\{\widetilde{W}_1\} \cdot \{D\})$, and the proof of the Lemma is over.

Remark 1.2. In particular, if we also choose $\{\varepsilon_1, \varepsilon_2\} \in H_1(E, \mathbf{Z})$ a symplectic basis, and the matrix of γ_* in the two given bases is

$$\gamma_* = \begin{pmatrix} a_{11} & \dots & a_{1g} & b_{11} & \dots & b_{1g} \\ a_{21} & \dots & a_{2g} & b_{21} & \dots & b_{2g} \end{pmatrix},$$

then it easily follows that

$$(1.2) \quad \deg(\gamma) = \sum_i \begin{vmatrix} a_{1i} & b_{1i} \\ a_{2i} & b_{2i} \end{vmatrix}.$$

Consider next a curve B of genus $g \geq 1$, E an elliptic curve, and $X \xrightarrow{\pi} B$ a principal elliptic bundle over B with fiber E (cf. [Kod], [Hf], [Bri3] for precise

definitions). If we suppose moreover, that $X \xrightarrow{\pi} B$ is not topologically trivial, then we have the following isomorphisms compatible with the cup-products (cf. [Bri1], [Bri2]):

$$(1.3) \quad H^2(X, \mathbf{Z})/\text{Tors}(H^2(X, \mathbf{Z})) \cong H^1(B, \mathbf{Z}) \otimes_{\mathbf{Z}} H^1(E, \mathbf{Z}),$$

and

$$(1.4) \quad NS(X)/\text{Tors}(NS(X)) \cong \text{Hom}(J_B, E^\vee),$$

where J_B denotes the Jacobian variety of B and E^\vee is the dual curve of E . The torsion of $H^2(X, \mathbf{Z})$ (as well as of $NS(X)$) is generated by the class of a fiber.

To avoid confusions, throughout the rest of the paper, for an element $c \in H^2(X, \mathbf{Z})$, we will denote by \hat{c} its class modulo $\text{Tors}(H^2(X, \mathbf{Z}))$.

We prove next a formula for computing the intersection form on $H^2(X, \mathbf{Z})$. It suffices, of course, to make it explicit for classes in $H^2(X, \mathbf{Z})/\text{Tors}(H^2(X, \mathbf{Z}))$. Consider then $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\} \subset H_1(B, \mathbf{Z})$ a symplectic basis with Kronecker duals $\alpha_1^*, \dots, \alpha_g^*, \beta_1^*, \dots, \beta_g^* \in H^1(B, \mathbf{Z})$, and $\{e_1, e_2\} \subset H^1(E, \mathbf{Z})$ a symplectic basis. Then any class $\hat{c} \in H^2(X, \mathbf{Z})/\text{Tors}(H^2(X, \mathbf{Z}))$ can be expressed as

$$\hat{c} = \sum_{k=1}^2 \sum_{i=1}^g (a_{ki}(\alpha_i^* \otimes e_k) + b_{ki}(\beta_i^* \otimes e_k)),$$

and the self-intersection computes as

$$\hat{c}^2 = -2 \sum_i (a_{1i}b_{2i} - b_{1i}a_{2i}).$$

Let now $\hat{c} \in NS(X)/\text{Tors}(NS(X))$. Choosing a base-point in B , we can think of the cohomology class \hat{c} as being a covering map $\hat{c}: B \rightarrow E$. By the considerations above, we get the following result (for a proof using the Riemann-Roch Theorem see [Te], Remark 1.11):

Lemma 1.3. *With the previous notations, $\hat{c}^2 = -2\deg(\hat{c})$.*

Suppose now that $C \xrightarrow{f} B$ is a covering; in this case, $Y = X \times_B C \rightarrow C$ is a principal elliptic bundle and $Y \rightarrow C$ is topologically non-trivial as soon as $X \rightarrow B$ is topologically non-trivial.

We give next a description of the push-forward map induced between the cohomology groups by the natural covering map $Y \xrightarrow{\varphi} X$. Firstly, let us remark that the class of a fiber in Y pushes-forward to the class of a fiber in X , therefore, we only need to describe the push-forward map φ_* induced between $H^2(Y, \mathbf{Z})/\text{Tors}(H^2(Y, \mathbf{Z}))$ and $H^2(X, \mathbf{Z})/\text{Tors}(H^2(X, \mathbf{Z}))$. Seeing the elements in $H^2(Y, \mathbf{Z})/\text{Tors}(H^2(Y, \mathbf{Z}))$ as maps from $H_1(C, \mathbf{Z})$ to $H_1(E^\vee, \mathbf{Z}) \cong H^1(E, \mathbf{Z})$, and identifying the pull-back morphism $f^*: J_B \rightarrow J_C$ with its rational representation, we have the following:

Lemma 1.4. *With the previous notations, $\varphi_*(\hat{c}) = \hat{c} \circ f^*$, for any class $\hat{c} \in H^2(Y, \mathbf{Z})/\text{Tors}(H^2(Y, \mathbf{Z}))$.*

Proof. The morphism $f^*: H_1(B, \mathbf{Z}) \rightarrow H_1(C, \mathbf{Z})$ is the pull-back morphism in homology via Poincaré duality. By using the canonical isomorphism (1.3), we get

for any element $z \in H_1(B, \mathbf{Z})$ the following equalities:

$$\varphi_*(\widehat{c})(z) = ((f_* \otimes id)(\widehat{c}))/z, \text{ and } (\widehat{c} \circ f^*)(z) = \widehat{c}/f^*(z),$$

where $"/$ is the *slant product*. Since f_* and f^* are transposed to each other, we conclude.

2. Vector bundles on primary Kodaira surfaces via coverings

Suppose that $r \geq 2$ is an integer, and $X \xrightarrow{\pi} B$ is a primary Kodaira surface over the elliptic curve B , with fiber E . Let C be a curve of genus 2, and let $C \xrightarrow{f} B$ be a (ramified) covering of degree r . Set $Y = X \times_B C \rightarrow C$, which is a principal elliptic bundle over C , and covers r -to-1 X by the natural map $Y \xrightarrow{\varphi} X$.

If one considers now a line bundle L on Y , the push-forward sheaf $\mathcal{E} = \varphi_* L$ is actually a rank- r vector bundle on X with

$$(2.1) \quad \begin{aligned} c_1(\mathcal{E}) &\equiv \varphi_* c_1(L) \text{ mod Tors}(NS(X)), \\ \Delta(\mathcal{E}) &= \frac{1}{2r^2} [(\varphi_* c_1(L))^2 - r c_1^2(L)]. \end{aligned}$$

(The push forward φ_* on cohomology is obtained as usual via Poincaré duality on both source and target space.)

Moreover, one can easily see that for any two line bundles L and L' on Y , the following holds:

$$(2.2) \quad c_1(\varphi_*(L \otimes L')) = c_1(\varphi_* L) + \varphi_* c_1(L').$$

If the covering f does not factor through an isogeny of B we can give *algebraic* interpretations for $c_1(\mathcal{E})$ and $\Delta(\mathcal{E})$ as follows. In this case, $F := \text{Ker}(J_C \rightarrow B)$ is an elliptic curve; consider $g : C \rightarrow F$ the complementary covering (see, for example [Ka2], [FrKa], [Mumf2]). Then the pull-back maps $f^* : B \rightarrow J_C$ and $g^* : F \rightarrow J_C$ turn out to be injective, and they give rise to an isogeny $f^* \times g^* : B \times F \rightarrow J_C$. Therefore, we can write $J_C \cong (B \times F)/H$, where $H = \text{Ker}(f^* \times g^*)$; the subgroup $H \subset (B \times F)[r]$ is in fact the graph of an isomorphism from $B[r]$ to $F[r]$, which is anti-isometric with respect to the Weil pairings.

Lemma 2.1. *In the hypotheses above, for any line bundle L on Y , $\mathcal{E} = \varphi_* L$ is a rank- r vector bundle on X with $\widehat{c_1(\mathcal{E})} = \widehat{c_1(L)} \circ f^*$ and $r^2 \Delta(\mathcal{E}) = \deg(\widehat{c_1(L)} \circ g^*)$.*

Proof. Let $\widehat{c} \in NS(Y)/\text{Tors}(NS(Y))$ and $\widehat{c}_1 \in NS(X)/\text{Tors}(NS(X))$ the classes associated to $c_1(L) \in NS(Y)$ and $c_1(\mathcal{E}) \in NS(X)$. Then Lemma 1.4 and the formulae (2.1) read $\widehat{c}_1 = \widehat{c} \circ f^*$ and

$$(2.3) \quad 2r^2 \Delta(\mathcal{E}) = \widehat{c}_1^2 - r \widehat{c}^2.$$

We have the following diagram:

$$\begin{array}{ccccc}
& & C & & \\
& g \swarrow & \downarrow j_C & \searrow f & \\
F & \xleftarrow{g_*} & J_C & \xrightarrow{f_*} & B \\
& \searrow g^* & \downarrow \hat{c} & \swarrow f^* & \\
& & E & &
\end{array}$$

where j_C is the embedding of C into its Jacobian J_C , $g = g_* \circ j_C$, $f = f_* \circ j_C$, $g_* \circ g^* = \text{rid}_F$, $f_* \circ f^* = \text{rid}_B$, $f_* \circ g^* = 0$ and $g_* \circ f^* = 0$.

Using Lemma 1.3 and (2.3) we see that the formula $r^2 \Delta(\mathcal{E}) = \deg(c_1(\widehat{L}) \circ g^*)$ is equivalent to:

$$(2.4) \quad r \deg(\hat{c} \circ j_C) = \deg(\hat{c} \circ f^*) + \deg(\hat{c} \circ g^*).$$

Let $\{W_1\}$, $\{f^*B\}$, $\{g^*F\} \in H^2(J_C, \mathbf{Z})$ be the classes of the divisors $j_C(C)$, f^*B and g^*F respectively, on the Jacobian J_C of C . Denote by S a fiber of the morphism $\hat{c} : J_C \rightarrow E$. Then we have:

$$\deg(\hat{c} \circ j_C) = (\{W_1\} \cdot \{S\}), \quad \deg(\hat{c} \circ f^*) = (\{f^*B\} \cdot \{S\}), \quad \deg(\hat{c} \circ g^*) = (\{g^*F\} \cdot \{S\}).$$

Now, consider the diagram:

$$\begin{array}{ccccc}
& & C & & \\
& g \swarrow & \downarrow q & \searrow f & \\
F & \longrightarrow & F \times B & \longleftarrow & B \\
& \searrow g^* & \downarrow p & \swarrow f^* & \\
& & J_C & &
\end{array}$$

where $q := (g, f)$, $p = g^* + f^*$, $p \circ q = j_C$, and the inclusions $F \hookrightarrow F \times B$, $B \hookrightarrow F \times B$ naturally identify F with $F \times \{0\}$ and B with $\{0\} \times B$. Since $p : F \times B \rightarrow J_C$ is a finite covering of degree r^2 , and since $p^*(W_1) \equiv r(F \times \{0\} + \{0\} \times B)$ (cf [Mumf2]; see also [Ka2]), we get the relations:

$$r^2 \deg(\hat{c} \circ f^*) = (\{p^*(f^*B)\} \cdot \{p^*S\}), \quad r^2 \deg(\hat{c} \circ g^*) = (\{p^*(g^*F)\} \cdot \{p^*S\}),$$

$$r^2 \deg(\hat{c} \circ j_C) = r(\{F \times \{0\} + \{0\} \times B\} \cdot \{p^*S\}).$$

Because $\{p^*(g^*F + f^*B)\} \cdot \{p^*S\} = r^2(\{F \times \{0\} + \{0\} \times B\} \cdot \{p^*S\})$, the conclusion follows.

3. Proof of the Theorem.

In the sequel, we shall use the following simple observation:

Remark 3.1. Let E and F be two elliptic curves and let $f : F \rightarrow E$ be an isogeny. If $\deg(f)$ and r are coprime, then the morphism induced between the r -torsion points, $f[r] : F[r] \rightarrow E[r]$ is an isomorphism.

Indeed, $\text{Ker}(f[r]) \subset F[r] \cap \text{Ker}(f)$ and the order of $F[r]$ and the order of $\text{Ker}(f)$ are both divisible by the order of $\text{Ker}(f[r])$. It follows $\text{Ker}(f[r]) = 0$, and thus $f[r]$ is an isomorphism.

Let us denote $R := r^2 \Delta(r, c_1, c_2) = rc_2 - (r-1)c_1^2/2$, and let d be the greatest common divisor of the integers r and R . The existence of a rank- r holomorphic vector bundle with Chern classes c_1 and c_2 , in the particular case $d = r$, was proved in [To2] by using unramified coverings of B with a suitable elliptic curve and deformations of sheaves. We can assume therefore that $d < r$, and we divide the proof of the Theorem in several steps.

Step 1. We reduce the proof to the case $d = 1$. Suppose $d > 1$. Since R is divisible by d , it follows that $c_1^2/2$ is divisible by d as well. By means of the Lemma in [To2], there exists an unramified covering of degree d , $t : B' \rightarrow B$ with a suitable elliptic curve B' such that, denoting by $\tilde{t} : X' := B' \times_B X \rightarrow X$ the canonically induced unramified covering, where $X' \xrightarrow{\pi'} B'$ is also a primary Kodaira surface, there exists a class $c'_1 \in NS(X')$ with $\tilde{t}^*(c_1) = dc'_1$. Set $r' = r/d$, $R' = R/d$, and $c'_2 = c_2 - (d-1)c_1'^2/2$. Then $R' = r'^2 \Delta(r', c'_1, c'_2)$ and R' and r' are coprime. If there exists a holomorphic rank- r' vector bundle \mathcal{E}' on the primary Kodaira surface X' with Chern classes $c_1(\mathcal{E}') = c'_1$ and $c_2(\mathcal{E}') = c'_2$, then we choose $\mathcal{E} := \tilde{t}_*(\mathcal{E}')$. A simple computation shows that \mathcal{E} is a rank- r holomorphic vector bundle on X with Chern classes $c_1(\mathcal{E}) = c_1$ and $c_2(\mathcal{E}) = c_2$.

Step 2. Suppose next $d = 1$ and consider a cyclic isogeny $\delta : F \rightarrow E$ of degree R , where F is a suitable elliptic curve. Let $\hat{c}_1 : B \rightarrow E$ be the isogeny induced by the class $c_1 \in NS(X)$. From Lemma 1.3 and Remark 3.1 we see that both morphisms $\delta[r] : F[r] \rightarrow E[r]$ and $\hat{c}_1[r] : B[r] \rightarrow E[r]$ are isomorphisms. Set $\psi := \delta[r]^{-1} \circ \hat{c}_1[r] : B[r] \xrightarrow{\cong} F[r]$. From the properties of the Weil pairings (cf. [Hu], 12.2.4) it follows that ψ is an anti-isometry. Now, we distinguish two cases:

Case (a). If ψ is irreducible, denote by $H_\psi = \text{Graph}(\psi) \subset (B \times F)[r]$. Weil's Theorem ([We] Satz 2; see also [Ka2], [FrKa]) ensures that the quotient $J_\psi := (B \times F)/H_\psi$ is the Jacobian J_C of a curve C of genus 2 which covers r -to-1 the elliptic curves B and F . Moreover, the morphism $\tilde{c} : B \times F \rightarrow E$ defined by $\tilde{c}(x, y) = \hat{c}_1(x) - \delta(y)$ for any pair $(x, y) \in B \times F$ factors through a morphism $\hat{c} : J_C = (B \times F)/H_\psi \rightarrow E$. It is clear that $\hat{c} \circ f^* = \hat{c}_1$ and $\hat{c} \circ g^* = \delta$.

We choose next a line bundle L on the principal elliptic bundle $Y := X \times_B C \rightarrow C$, whose Chern class modulo $\text{Tors}(NS(Y))$ equals \hat{c} . If $\varphi : Y \rightarrow X$ denotes the corresponding covering of degree r , then Lemma 2.1 precisely says that $\mathcal{E} := \varphi_* L$ is a holomorphic rank- r vector bundle with $c_1(\mathcal{E}) \equiv c_1 \text{ modulo } \text{Tors}(NS(X))$ and discriminant $\Delta(\mathcal{E}) = \deg(\delta)/r^2 = \Delta(r, c_1, c_2)$.

Case (b). If ψ is reducible, we use the "Reducibility criterion" from [Ka2]. Denote again $H_\psi = \text{Graph}(\psi) \subset (B \times F)[r]$ and take the quotient $J_\psi = (B \times F)/H_\psi$. The morphism $\tilde{c} : B \times F \rightarrow E$ defined by $\tilde{c}(x, y) = \hat{c}_1(x) - \delta(y)$ for any pair $(x, y) \in B \times F$

factors through a morphism $\widehat{c} : J_\psi \rightarrow E$. This time J_ψ is no longer the Jacobian of a curve of genus 2. Since ψ is reducible, we get the so-called “diamond configuration” (cf. [Ka2]), i.e. there exist two elliptic curves E_1 and E_2 , an integer k with $1 \leq k < r$, and isogenies $h : B \rightarrow F$, $f_i : B \rightarrow E_i$, $f'_i : E_i \rightarrow F$, for $i = 1, 2$, such that $h = f'_1 \circ f_1 = f'_2 \circ f_2$, $\deg(f_1) = \deg(f'_2) = r - k$, $\deg(f'_1) = \deg(f_2) = k$. Moreover, if $\widetilde{f'_i} : F \rightarrow E_i$ denotes the dual map of f'_i , for $i = 1, 2$, then J_ψ is isomorphic to $E_1 \times E_2$ via the map $p : J_\psi \rightarrow E_1 \times E_2$, $p(\widehat{x, y}) := (f_1(x) - \widetilde{f'_1}(y), f_2(x) + \widetilde{f'_2}(y))$.

The natural inclusion of B in $E_1 \times E_2$ is given by (f_1, f_2) and the natural inclusion of F in $E_1 \times E_2$ is given by $(-\widetilde{f'_1}, \widetilde{f'_2})$. It follows easily that $\widehat{c} \circ (f_1, f_2) = \widehat{c}_1$ and $\widehat{c} \circ (-\widetilde{f'_1}, \widetilde{f'_2}) = -\delta$. We consider the maps $c' : E_1 \rightarrow E$ and $c'' : E_2 \rightarrow E$ given by $c'(u) = \widetilde{c}(u, 0)$, and $c''(v) = \widetilde{c}(0, v)$, respectively. Then a simple computation gives $\widehat{c}_1 = c' \circ f_1 + c'' \circ f_2$. Moreover, composing the relation $\widehat{c} \circ (-\widetilde{f'_1}, \widetilde{f'_2}) = -\delta$ by h to the right we get:

$$(3.1) \quad k(c' \circ f_1) - (r - k)(c'' \circ f_2) = \delta \circ h.$$

Consider now the unramified coverings $\widetilde{f}_1 : E_1 \rightarrow B$ and $\widetilde{f}_2 : E_2 \rightarrow B$ of degree $(r - k)$ and k respectively. Let $X_1 \rightarrow E_1$ and $X_2 \rightarrow E_2$ be the primary Kodaira surfaces defined by these coverings by taking the fibered products. Choose line bundles L_1 and L_2 on X_1 respectively on X_2 , with Chern classes modulo torsion c' and c'' respectively. Denote $X_1 \xrightarrow{\varphi_1} X$ and $X_2 \xrightarrow{\varphi_2} X$ the induced unramified coverings of degrees $(r - k)$, respectively k , and take the holomorphic vector bundles on X , $\mathcal{E}_1 := \varphi_{1,*}(L_1)$, and $\mathcal{E}_2 := \varphi_{2,*}(L_2)$, of ranks $(r - k)$, respectively k . These vector bundles have Chern classes (modulo torsion) $c' \circ f_1$, respectively, $c'' \circ f_2$, and vanishing discriminants. Finally, set $\mathcal{E} := \mathcal{E}_1 \oplus \mathcal{E}_2$. Then \mathcal{E} is a rank- r vector bundle with $c_1(\mathcal{E}) \equiv c_1$ modulo $\text{Tors}(NS(X))$. Now, using (3.1) a simple computation leads us to :

$$\Delta(\mathcal{E}_1 \oplus \mathcal{E}_2) = \frac{1}{2r} \left[\frac{\widehat{c}_1^2}{r} - \frac{(c' \circ f_1)^2}{r - k} - \frac{(c'' \circ f_2)^2}{k} \right] = \Delta(r, c_1, c_2).$$

Step 3. In order to get rid of the torsion, we need to add multiples of a class of a fiber. To do this, in the irreducible case, we consider F_Y to be a fiber of the projection map $Y \rightarrow C$, F_X to be a fiber of the projection map $X \rightarrow B$, and for any $m \geq 0$, we set $L_m = \mathcal{O}_Y(mF_Y)$. In homology, $\varphi_*\{F_Y\} = \{F_X\}$, and thus $\varphi_*(c_1(L_m)) = mc_1(\mathcal{O}_X(F_X))$. Formula (2.2) reads here $c_1(\varphi_*(L \otimes L_m)) = c_1(\mathcal{E}) + mc_1(\mathcal{O}_X(F_X))$. In the reducible case, we apply a similar argument, using a fiber of $X_1 \rightarrow X$ or of $X_2 \rightarrow X$, and this ends the proof.

Remark 3.2. For a primary Kodaira surface the compactness theorem in [To3] (Theorem 5.9) combined with the existence result above, produces moduli spaces of stable bundles, which are non-empty, holomorphically symplectic compact manifolds, when the Chern classes are chosen in the stably irreducible range as in [To3]. For example, if $c_1 \in NS(X)$ is chosen such that $m(2, c_1) > 0$ and $0 \leq \Delta(2, c_1, c_2) < m(2, c_1)$, we are in this range and all the 2-vector bundles with the given invariants are stable with respect to any Gauduchon metric on the primary Kodaira surface.

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